F (4)/Spin(4) Gauge Symmetry of Quantum Space-Time Dynamics.

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(Dated:)

The exceptional lie group F(4) and it's lie algebra offered me an intriguing possibility. Associated with F(4) is a four dimensional body centered cubic lattice. In my work on quantum gravity I have gone to pains to proclaim that space time is not a lattice of fixed points. All my studies still indicate that is true. However F(4) and it's "lattice" seem to have the symmetry of quantum space-time. Using the usual techniques of quantum field theory I have investigated this symmetry. As is true of any symmetry F(4)/Spin(4) symmetry, by way of Noether's theorem imposes a conserved current. A current which has the dimensions of length and contains the operator in my unpublished theory of quantum space-time dynamics gives space-time intervals. This means that space-time is a conserved quantity. The implications of this finding for cosmology are briefly considered.

The Real Representation of F(4)/Spin(4)

The Lie group F(4) is the isometry group of a 16 dimensional riemannian manifold. It can be constructed by adding 16 spinors to SO(9). This would be too much for modeling quantum space-time. What needs to be modded out of the group are the components of Spin(4). This is so for physical reasons as Spin(4) contains transformations that denote rotating one coordinate axis into another. Which really ammounts to a redefinition of coordinate axes which should have no physical consequences. That physics is invariant under such transformations is contained in the local Lorentz symmetry of the theory. After moding out that we are left with a 46 dimensional lie group. F(4)/Spin(4) which I will denote F(4)/Spin(4).

The following are the simplest matricies that will represent F(4)/Spin(4) they are based on it's root vectors. Using the plus-minus symbol I can write all of the possiblities (i.e. If one writes +/- 1 that is really two things -1 and +1. The same concept is used below to write all 46 matricies without haaving to write 46 matricies.)

$$\boldsymbol{\alpha} = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{\{\pm 1, 0, 0, 0\}, \{0, \pm 1, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}$

$$\boldsymbol{\beta} = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{\{\pm 1, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, \pm 1, 0\}, \{0, 0, 0, 0\}\}$

$$\boldsymbol{\gamma} = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}$$

$\{\{\pm 1, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, \pm 1\}\}$

$$\boldsymbol{\delta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{\{0, 0, 0, 0\}, \{0, \pm 1, 0, 0\}, \{0, 0, \pm 1, 0\}, \{0, 0, 0, 0\}\}$

$$\boldsymbol{\epsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}$$

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$\{\{0,0,0,0\},\{0,\pm1,0,0\},\{0,0,0,0\},\{0,0,0,\pm1\}\}$

$$\zeta = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \pm\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \pm\mathbf{1} \end{pmatrix}$$

$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,\pm1,0\},\{0,0,0,\pm1\}\}$

$$\eta = \begin{pmatrix} \pm\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$\{\{\pm1,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}$

$$\theta = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \pm\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$\{\{0,0,0,0\},\{0,\pm1,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}$

$$\kappa = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \pm\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,\pm1,0\},\{0,0,0,0\}\}$

$$\lambda = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \pm\mathbf{1} \end{pmatrix}$$

$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,\pm1\}\}$

$$\mu = \begin{pmatrix} \pm\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \pm\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \pm\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \pm\mathbf{1} \end{pmatrix}$$

$\{\{\pm1,0,0,0\},\{0,\pm1,0,0\},\{0,0,\pm1,0\},\{0,0,0,\pm1\}\}$

$$\Theta = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}$

I claim that this is a representation of the real valued variant of F (4)/Spin(4) as well as a basis for the space underlying the Lie algebra of F (4)/Spin(4). The simplest way to do this is to compute the multiplication table for the group. Right off the bat I can say that the table will be symmetric due to the fact that all of the elements of this representation are diagonal matrices. For that reason I will not bother to compute them independently.

$\alpha.\alpha$	$\alpha.\beta$	$\alpha.\gamma$	$\alpha.\delta$	$\alpha.\epsilon$	$\alpha.\zeta$	$\alpha.\eta$	$\alpha.\theta$	$\alpha.\kappa$	$\alpha.\lambda$	$\alpha.\mu$	$\alpha.\nu$	$\alpha.\xi$	$\alpha.\sigma$	$\alpha.\rho$	$\alpha.\Theta$
□	$\beta.\beta$	$\beta.\gamma$	$\beta.\delta$	$\beta.\epsilon$	$\beta.\zeta$	$\beta.\eta$	$\beta.\theta$	$\beta.\kappa$	$\beta.\lambda$	$\beta.\mu$	$\beta.\nu$	$\beta.\xi$	$\beta.\sigma$	$\beta.\rho$	$\beta.\Theta$
□	□	$\gamma.\gamma$	$\gamma.\delta$	$\gamma.\epsilon$	$\gamma.\zeta$	$\gamma.\eta$	$\gamma.\theta$	$\gamma.\kappa$	$\gamma.\lambda$	$\gamma.\mu$	$\gamma.\nu$	$\gamma.\xi$	$\gamma.\sigma$	$\gamma.\rho$	$\gamma.\Theta$
□	□	□	$\delta.\delta$	$\delta.\epsilon$	$\delta.\zeta$	$\delta.\eta$	$\delta.\theta$	$\delta.\kappa$	$\delta.\lambda$	$\delta.\mu$	$\delta.\nu$	$\delta.\xi$	$\delta.\sigma$	$\delta.\rho$	$\delta.\Theta$
□	□	□	□	$\epsilon.\epsilon$	$\zeta.\epsilon$	$\eta.\epsilon$	$\theta.\epsilon$	$\kappa.\epsilon$	$\lambda.\epsilon$	$\mu.\epsilon$	$\nu.\epsilon$	$\xi.\epsilon$	$\sigma.\epsilon$	$\rho.\epsilon$	$\Theta.\epsilon$
□	□	□	□	□	$\zeta.\zeta$	$\zeta.\eta$	$\theta.\zeta$	$\kappa.\zeta$	$\lambda.\zeta$	$\mu.\zeta$	$\nu.\zeta$	$\xi.\zeta$	$\sigma.\zeta$	$\rho.\zeta$	$\Theta.\zeta$
□	□	□	□	□	□	$\eta.\eta$	$\theta.\eta$	$\kappa.\eta$	$\lambda.\eta$	$\mu.\eta$	$\nu.\eta$	$\xi.\eta$	$\sigma.\eta$	$\rho.\eta$	$\Theta.\eta$
□	□	□	□	□	□	□	$\theta.\theta$	$\kappa.\theta$	$\lambda.\theta$	$\mu.\theta$	$\nu.\theta$	$\xi.\theta$	$\sigma.\theta$	$\rho.\theta$	$\Theta.\theta$
□	□	□	□	□	□	□	□	$\kappa.\kappa$	$\lambda.\kappa$	$\mu.\kappa$	$\nu.\kappa$	$\xi.\kappa$	$\sigma.\kappa$	$\rho.\kappa$	$\Theta.\kappa$
□	□	□	□	□	□	□	□	□	$\lambda.\lambda$	$\mu.\lambda$	$\nu.\lambda$	$\xi.\lambda$	$\sigma.\lambda$	$\rho.\lambda$	$\Theta.\lambda$
□	□	□	□	□	□	□	□	□	□	$\mu.\mu$	$\nu.\mu$	$\xi.\mu$	$\sigma.\mu$	$\rho.\mu$	$\Theta.\mu$
□	□	□	□	□	□	□	□	□	□	□	$\nu.\nu$	$\xi.\nu$	$\sigma.\nu$	$\rho.\nu$	$\Theta.\nu$
□	□	□	□	□	□	□	□	□	□	□	□	$\xi.\xi$	$\sigma.\xi$	$\rho.\xi$	$\Theta.\xi$
□	□	□	□	□	□	□	□	□	□	□	□	□	$\sigma.\sigma$	$\rho.\sigma$	$\Theta.\sigma$
□	□	□	□	□	□	□	□	□	□	□	□	□	□	$\rho.\rho$	$\Theta.\rho$
□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	$\Theta.\Theta$

//MatrixForm

I have suppressed this output as it runs off the page. Anyone who wishes to see it should email me. I will send the Mathematica Notebook file for your perusal.

By examination of this output it can be seen that these matrices form a group under matrix multiplication. This is not hard to check mentally take any two of the matricies I give above and multiply them together and you get another matrix in the group. They are 4x4 and real valued. As a group they are simple. This group also represents a continuous symmetry of the body centered cubic lattice and as such this is a lie group. Therefore this must be a real representation of the lie group F(4)/Spin(4). This will be the representation I will use in this composition.

The Lie Algebra F(4)/Spin(4)

Does F(4)/Spin(4) have an associated Lie Algebra?

A Lie algebra is a vector space L defined over the real numbers with a binary operation [,]that satisfies the following axioms. Where X and Y are in L.

1. $[(\mathrm{X1}+\mathrm{X2}),Y]=[\mathrm{X1},Y]+[\mathrm{X2},Y]$
2. $[(\mathrm{qX}),Y]=q[X,Y] \quad q\in\mathbb{C}$.
3. $[X,Y]=-[Y,X]$
4. $[X,[Y,Z]]+[Z,[X,Y]]+[Y,[X,Z]]=0$

Take as L the vector space of diagonal 4x4 matrices with the basis for the space being $\{\eta,\theta,\kappa,\lambda\}$. Those being 4x4 matrices with a 1 in one of the diagonal positions. Any diagonal matrix can be written in terms of those matrices. Then take the Lie bracket as being the commutator.

The fourth axiom is satisfied by the commutativity of all diagonal matrices as is axiom 3. The linearity of the commutator and matrices guarantees that the first and second axioms are also satisfied.

Because these axioms are satisfied I can say that this is a lie algebra. This empowers me to use F(4)/Spin(4) to investigate further.

The Gauge Field With the Symmetry of F(4)/Spin(4).

So what would a field with the gauge symmetry of F(4)/Spin(4) look like? What would a Lagrangian have to look like for it to be invariant under a F(4)/Spin(4) transformation?

Before I answer these questions I want to make an observation about the structure of F(4)/Spin(4). Any element of F(4)/Spin(4) can be written in terms of four of the matrices in the representation used in this paper. Specifically $\{\eta,\theta,\kappa,\lambda\}$. Taken together these can form a soft of pseudo four vector.

$F_\lambda=\mathrm{a}\eta+\mathrm{b}\theta+\mathrm{c}\kappa+\mathrm{d}\lambda$

This is true because all of the matrices in this representation are diagonal and they have one's or zero's on the diagonal. The elements of those four matrices in the representation of F(4)/Spin(4) have just a one on the diagonal and zero's elsewhere. Therefore they form a basis for the space of possible transformations in F(4)/Spin(4) as represented by these matrices. In more technical jargon I should say that the matricies $\{\eta,\theta,\kappa,\lambda\}$ along with $+$ generate the group F(4)/Spin(4). This construction would also have all the same algebra as the matrices in F(4)/Spin(4) do.

So what field would be symmetric under this transformation? it could *not* be a vector (or 1-form) field recall the motivation of this paper is the observation that the F4 lattice has the same structure and symmetry one would expect for Planck-scale space-time, or quantum gravity. It is known from General relativity that gravity must be a two form, or tensor field. It is also known from most thinking on the nature of the supposed graviton that it would be a spin-2 boson not a spin 1 or vector (1-form) particle. *Therefore* only tensors need to be considered.

Consider the stress energy tensor $T_{\alpha\beta}$. Of all the fields in my theory it is the most physical and unambiguously defined.

$T_{\alpha\beta}\rightarrow F_\lambda T_{\alpha\beta}F^{\lambda+}$

Where the $+$ super script indicates the pseudo inverse of the matrix. A pseudo inverse has to be used because these matrices are singular.

Working through this the field in my theory and the field of empty space-time in classical general relativity are diagonal. The fields F_λ are diagonal. Because of the high degree of symmetry that exist in diagonal matrices we can shuffle these fields around at will. So what I will do is re write the last equation like so.

$T_{\alpha\beta}\rightarrow F_\lambda T_{\alpha\beta}F^{\lambda+}\rightarrow T_{\alpha\beta}F_\lambda F^{\lambda+}$

By the usual rules of the pseudo inverse I could re write the last F's as just one F. Instead I will use the fact that a pseudo inverse in effect as well as Einstein summation. Each of those λ's is a matrix and each of those matrices is a zero with a one at a diagonal. Therefore $F_\lambda F^{\lambda+} \rightarrow$I or the identity matrix. Therefore I can write that

$T_{\alpha\beta} \rightarrow F_\lambda T_{\alpha\beta} F^{\lambda+} \rightarrow T_{\alpha\beta} F_\lambda F^{\lambda+} \rightarrow T_{\alpha\beta} I \rightarrow T_{\alpha\beta}$

Therefore any diagonal matrix representation of these fields would have F(4)/Spin(4) symmetry... However there is only one field that has to be diagonal in any basis for physical reasons. The stress energy tensor of general relativistic space-time. It's matrix rep has to have only diagonal entries just like that of an ideal fluid.

Therefore I say that this field of quantized space time is THE field which is invariant under the F(4)/Spin(4) transformation. This may sound a bit odd to at first until one considers that the classical Einstein equation in a mass free region is a simple scaling relationship between the Einstein tensor and the stress energy tensor. $G^{\mu\nu}=8\pi T^{\mu\nu}$.

The Lagrangian

The next step is to write the Lagrangian for this field. The question is this: what is the simplest Lagrangian that can be made out of the given mathematical objects in this theory? First consider what the free field Lagrangian would look like.

In gravity what would happen is the non gravitational stress energies propagate through what they see as locally flat space time. But as they propagate along and as they available geodetic path's vary they are pulled along through to their possible futures. These possible futures would be given by the S tensor in my theory. $S^{\mu\nu}$ Therefore I propose that a term in this Lagrangian should be of the form $\Delta_\lambda S^{\alpha\beta} T_{\alpha\beta}$. ($\Delta_\lambda$ is a covariant quantum derivative see**?**]) but this form has the problem of not being a Lorentz scalar. If I use the f matrices just as the γ 's are used in QED I can make it a Lorentz scalar.

$L = f^\lambda \triangle_\lambda S^{\alpha\beta} T_{\alpha\beta} = f^\lambda \triangle_\lambda @_\delta^\alpha T^{\delta\beta} T_{\alpha\beta}$

Now I will consider the interaction term. Thinking physically what happens in gravitation is one particle interacts with another particle by exchanging a graviton. In this case I shall use the A tensor and the @ tensor.

$L_{\text{int}} = -\sqrt{G} G A^{\alpha\delta} @_\delta{}^\gamma A_{\alpha\gamma}$

Last but not least there is the fact that we have a field with an Abelian field with a lie algebra and it's symmetry. We know from experience with QED that a gauge invariant field needs to be introduced.

$F_\alpha^{\delta\beta} = \left(\triangle_\alpha A^{\delta\beta} - \triangle_\delta A^{\alpha\beta}\right)$

Now the total Lagrangian can be written.

$L = f^\lambda \triangle_\lambda @_\delta^\alpha T^{\delta\beta} T_{\alpha\beta} - \sqrt{G} G \left(\frac{1}{4} F_\alpha{}^{\delta\beta} F^\alpha{}_{\delta\beta} - A^{\alpha\delta} @_\delta^\gamma A_{\alpha\gamma}\right)$

Conserved Current

My next task is to elevate F(4)/Spin(4) to a local symmetry. This is not difficult. All one needs to do is consider a single element of the basis for the space F(4)/Spin(4). Like so

$T_{\alpha\beta} \rightarrow f T_{\alpha\beta} f^+ \rightarrow f f^+ T_{\alpha\beta} \rightarrow f T_{\alpha\beta}$

$T_{\alpha\beta} \rightarrow f T_{\alpha\beta}$ Is by the nature of this theory both discrete and continuous. I say this because this theory is formulated to be most valid at the Planck scale which theoretically is the smallest meaningful length that can be. So to transform by such an increment meets both definitions.

Like all continuous symmetries of the action this one will also have a conserved current. Figuring this out is elementary to QFT since the Lagrangian is known. The conserved current is

$J^{\alpha\beta}=f^\lambda \; @^\alpha{}_\lambda \; f_\delta T^{\delta\beta}$

Looking at the dimensions this has the units of length of length and the operator $S^{\alpha\beta} = @^\alpha{}_\lambda T^{\lambda\beta}$ found in Quantum Space-Time Dynamics. This current is of space-time and the conserved quantity is in fact space-time.

Conclusions

Because each piece of this Lagrangian density has been shown to be invariant under F(4)/Spin(4) gauge transformation then I have found the Lagrangian of the field that is invariant under the F(4)/Spin(4) symmetry. Moreover I have shown by induction that this Lagrangian is the correct Lagrangian for Quantum Space-Time Dynamics.

Furthermore I have shown that the conserved current in this theory is none other than the "flow" of space-time. The consequences of this are that space-time cannot be created or destroyed and that it is in finite supply. This result will have a great impact on cosmology and physics. This result troubled me at first as it seems counter intuitive to think that space-time is not simply a void into which our universe expanded after the big bang. In fact this conservation law says that the total amount of space time that there is finite, and that space-time cannot be destroyed or created. As counter intuitive as it is on first sight on second sight it makes perfect sense. So far as I know there is no interaction that creates or destroys space-time. There are interactions that bend and fold it but none that create or destroy it. Even a black hole merely warps space-time, and by current black hole theory, space-time returns to normal once the hole evaporates. Therefore space-time itself must be a conserved quantity.

Another conjecture based on this analysis can be made about the nature of the big bang. Cosmology has struggled with what banged and why. Well this theory provides a partial answer. What banged was not a infinitely dense point. But a little cell of space-time which had perfect F(4)/Spin(4) symmetry and was of Planck scale dimensions. The forces were all unified within it. Then for some reason that symmetry broke for just an instant. In that instant all of the space time that there will ever be was created and is unfolding. Space-time is unfolding and seeking a state of lowest energy or least curvature (same thing really).

In a less cosmological sense this result, that space-time, is a conserved current means that no interaction that does not break the F(4)/Spin(4) symmetry can change the length of a geodetic path. They can bend and twist but their total length is fixed. Basically unless acted upon by an outside force an object will just proceed on it's geodetic path, this is simply the law of inertia in a new context. One where it has been derived from first principles and not simply assumed at the outset. In the process a simple field theoretic construction of Quantum Gravity has been revealed.

- [] "The Dynamics of Planck Scale Space-Time `http://www2.uic.edu/~hfarme2/TDPSST.xml`", H. Farmer (Unpublished body of work on quantum gravity, and geometry at the Planck Scale). http://www2.uic.edu/~hfarme2/TDPSST.xml
- [] "The Octonians `http://math.ucr.edu/home/baez/octonions/node15.html`", J. Baez, Bull. Amer. Math. Soc., 39 (2002), 145-205.,Posted: December 21, 2001
- [] "An Introduction to Quantum Field Theory", M. Peskin, D. Schroeder, Westview press, ISBN 0-201-50397-2, Pages 17 - 19